WEEKLY TEST OYJ SOLUTION MATHMEATICS 30 JUNE 2019

31. (b) $by^2 = (x+a)^3 \Rightarrow 2by.\frac{dy}{dx} = 3(x+a)^2 \Rightarrow \frac{dy}{dx} = \frac{3}{2by}(x+a)^2$

$\therefore$ Subnormal $= y\frac{dy}{dx} = \frac{3}{2b}(x+a)^2$

$\therefore$ Subtangent $= \frac{y}{\left(\frac{dy}{dx}\right)} = \frac{y}{\frac{3(x+a)^2}{2by}} = \frac{2by^2}{3(x+a)^2}$

$= \frac{2b\frac{(x+a)^3}{b}}{3(x+a)^2} = \frac{2}{3}(x+a)$

$\therefore$ (Subtangent)^2 = $\frac{4}{9}(x+a)^2$

and $\frac{(\text{Subtangent})^2}{\text{Subnormal}} = \frac{\frac{4}{9}(x+a)^2}{\frac{3}{2b}(x+a)^2} = \frac{8b}{27}$

$\Rightarrow$ (Subtangent)^2 = constant $\times$ (Subnormal).

$\therefore$ (Subtangent)$^2 \propto$ (Subnormal).

32. (a) $\sqrt{x} + \sqrt{y} = a$; $\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$, $\therefore$ $\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$

Hence tangent at (x, y) is $Y - y = -\frac{\sqrt{y}}{\sqrt{x}}(X - x)$

or $X\sqrt{y} + Y\sqrt{x} = \sqrt{xy}\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{axy}$

or $\frac{X}{\sqrt{a}\sqrt{x}} + \frac{Y}{\sqrt{a}\sqrt{y}} = 1$.

Clearly its intercepts on the axes are $\sqrt{a}\sqrt{x}$ and $\sqrt{a}\sqrt{y}$.

Sum of the intercepts = $\sqrt{a}\left(\sqrt{x} + \sqrt{y}\right) = \sqrt{a}.\sqrt{a} = a$.

33. (b) Clearly the point of intersection of curves is (0, 1). Now, slope of tangent of first curve $m_1 = \frac{dy}{dx} = a^x \log a$

$\Rightarrow \left(\frac{dy}{dx}\right)_{(0,1)} = m_1 = \log a$

Slope of tangent of second curve $m_2 = \frac{dy}{dx} = b^x \log b$

$\Rightarrow m_2 = \left(\frac{dy}{dx}\right)_{(0,1)} = \log b$

$\therefore \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\log a - \log b}{1 + \log a \log b}$.

34. (a) Given $y = 6x - x^2$ .....(i)

$\frac{dy}{dx} = 6 - 2x$

Since, tangent is parallel to the line $4x - 2y - 1 = 0$

$\therefore \frac{dy}{dx} = 6 - 2x = \frac{-4}{-2} \Rightarrow 6 - 2x = 2 \Rightarrow x = 2$

Put the value of x in (i), we get $y = 8$

Hence required point of tangency will be $(2, 8)$.

35. (a) $\frac{dx}{d\theta} = a(1 + \cos\theta), \frac{dy}{d\theta} = a(\sin\theta)$

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{2}} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{a\sin\theta}{a(1+\cos\theta)} = 1, \quad y|_{\theta=\frac{\pi}{2}} = a$$

Length of sub-tangent $ST = \frac{y}{dy/dx} = \frac{a}{1} = a.$

and length of sub-normal $SN = y\frac{dy}{dx} = a.1 = a$

Hence $ST = SN$.

36. (c) $x|_{\theta=\frac{\pi}{4}} = \frac{2}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$,

$$y|_{\theta=\frac{\pi}{4}} = \frac{3}{2\sqrt{2}}, \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} \left.\frac{9\sin^2\theta\cos\theta}{-6\cos^2\theta\sin\theta}\right|_{\theta=\frac{\pi}{4}} = \frac{-3}{2}.$$

$\therefore$ Equation of tangent is $\left(y - \frac{3}{2\sqrt{2}}\right) = \frac{-3}{2}\left(x - \frac{1}{\sqrt{2}}\right)$

$\Rightarrow 3\sqrt{2}x + 2\sqrt{2}y = 6 \Rightarrow 3x + 2y = 3\sqrt{2}$.

37. (b) Curve $x + y = e^{xy}$

Differentiating with respect to x

$$1 + \frac{dy}{dx} = e^{xy}\left(y + x\frac{dy}{dx}\right) \text{ or } \frac{dy}{dx} = \frac{ye^{xy} - 1}{1 - xe^{xy}}$$

$$\frac{dy}{dx} = \infty \Rightarrow 1 - xe^{xy} = 0 \Rightarrow 1 - x(x + y) = 0$$

This hold for $x = 1$, $y = 0$.

38. (b) We have $f(x)g(x) = 1$

Differentiating with respect to x, we get

$$f'g + fg' = 0 \quad \text{.....(i)}$$

Differentiating (i) *w.r.t.* x, we get

$$f''g + 2f'g' + fg'' = 0 \quad \text{.....(ii)}$$

Differentiating (ii) *w.r.t.* x, we get

$$f'''g + g''' f + 3f''g' + 3g''f' = 0$$

$$\Rightarrow \frac{f'''}{f'}(f'g) + \frac{g'''}{g'}(fg') + \frac{3f''}{f}(fg') + \frac{3g''}{g}(gf') = 0$$

$$\Rightarrow \left(\frac{f'''}{f'} + \frac{3g''}{g}\right)(f'g) = -\left(\frac{g'''}{g'} + \frac{3f''}{f}\right)(fg')$$

$$\Rightarrow -\left(\frac{f'''}{f'} + \frac{3g''}{g}\right)(fg') = -\left(\frac{g'''}{g'} + \frac{3f''}{g}\right)fg', \text{ [using (i)]}$$

$$\Rightarrow \frac{f'''}{f'} + \frac{3g''}{g} = \frac{g'''}{g'} + \frac{3f''}{f} \Rightarrow \frac{f'''}{f'} - \frac{g'''}{g'} = 3\left(\frac{f''}{f} - \frac{g''}{g}\right).$$

39. (d) $I_n = \frac{d^{n-1}}{dx^{n-1}}[x^{n-1} + nx^{n-1}\log x]$

$I_n = (n-1)! + nI_{n-1} \Rightarrow I_n - nI_{n-1} = (n-1)!$.

40. (a) Given that $x = \sin t$, $y = \sin pt$

$$\frac{dx}{dt} = \cos t, \ \frac{dy}{dt} = p\cos pt$$

$\therefore\ \frac{dy}{dx} = \frac{p\cos pt}{\cos t} = \frac{p\sqrt{1-y^2}}{\sqrt{1-x^2}}$

Again differentiate *w.r.t.* x,

$$\frac{d^2y}{dx^2} = \frac{p\sqrt{1-x^2}.\frac{1}{2\sqrt{1-y^2}}.(-2y)\frac{dy}{dx} - p\sqrt{1-y^2}.\frac{1}{2\sqrt{1-x^2}}(-2x)}{(1-x^2)}$$

$$(1-x^2)\frac{d^2y}{dx^2} = -py\frac{\sqrt{1-x^2}}{\sqrt{1-y^2}}\frac{dy}{dx} + px\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

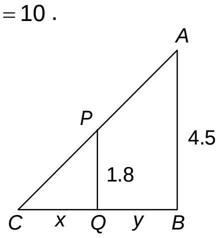
$$(1-x^2)\frac{d^2y}{dx^2} = -p^2y + x\frac{dy}{dx}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + p^2y = 0.$$

41. (c) Here $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$

$\Rightarrow\ \frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}\ \Rightarrow\ \frac{dr}{dt} = \frac{40}{4\pi r^2} = \frac{5}{32\pi}$

$\therefore \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi \times 8 \times \frac{5}{32\pi} = 10$.



42. (a) Given curve $y^2 = px^3 + q$ .....(i)

Differentiate with respect to x, $2y.\frac{dy}{dx} = 3px^2$

$\Rightarrow\ \frac{dy}{dx} = \frac{3p}{2}\left(\frac{x^2}{y}\right)$

$\therefore \left|\frac{dy}{dx}\right|_{2,3} = \frac{3p}{2} \times \frac{4}{3} = 2p$

For given line, slope of tangent $= 4$

$\therefore 2p = 4\ \Rightarrow\ p = 2$

From equation (i), $9 = 2 \times 8 + q\ \Rightarrow q = -7$.

43. (a) $y = \frac{2}{3}x^3 + \frac{1}{2}x^2\ \Rightarrow\ \frac{dy}{dx} = 2x^2 + x$ .....(i)

Now tangent makes equal angle with axis

$\therefore\ y = 45^o$ or -45^o

$\therefore\ \frac{dy}{dx} = \tan(\pm 45^o) = \pm\tan(45^o) = \pm 1$

$\therefore$ From equation (i), $2x^2 + x = 1$ (taking $+ve$ sign)

$\Rightarrow\ 2x^2 + x - 1 = 0\ \Rightarrow\ (2x-1)(x+1) = 0$

$\therefore x = \frac{1}{2}, -1$

From the given curve, when $x=\frac{1}{2}$, $y=\frac{2}{3}.\frac{1}{8}+\frac{1}{2}.\frac{1}{4}=\frac{1}{12}+\frac{1}{8}=\frac{5}{24}$ and when $x=-1$, $y=\frac{2}{3}(-1)+\frac{1}{2}.1$ =

$-\frac{2}{3}+\frac{1}{2}=-\frac{1}{6}$

Therefore, required points are $\left(\frac{1}{2},\frac{5}{24}\right)$ and $\left(-1,-\frac{1}{6}\right)$.

44. (d) Slope of the normal = $\frac{-1}{dy/dx}$

$\Rightarrow \tan\frac{3\pi}{4}=\frac{-1}{(dy/dx)_{(3,4)}}$

$\therefore \left(\frac{dy}{dx}\right)_{(3,4)}=1$, $\therefore f'(3)=1$.

45. (d) $y^3+3x^2=12y \Rightarrow 3y^2.\frac{dy}{dx}+6x=12.\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx}(3y^2-12)+6x=0 \Rightarrow \frac{dy}{dx}=\frac{6x}{12-3y^2}$

$\Rightarrow \frac{dx}{dy}=\frac{12-3y^2}{6x}$

Since tangent is parallel to y-axis

$\therefore \frac{dx}{dy}=0 \Rightarrow 12-3y^2=0$ or $y=\pm 2$.

Then $x=\pm\frac{4}{\sqrt{3}}$. At $\left(\pm\frac{4}{\sqrt{3}},-2\right)$; the equation of curve doesn't satisfy.